Kenmerk: TW13/ST-???
Datum: 30 oktober 2013

Universiteit Twente

Probability theory (191530062)
Donderdag 31 oktober 2013 van 8.45-11.45 uur

This exam consists of 7 exercises and a table of the standard normal distibution.
Using a *simple* calculator is allowed, a *graphic* calculator is *not* allowed.
Put name and studentnummer on all your submitted work.

Note: Indicate whether you do **only the second test** (part 2, exercises 4,5,6,7), or **the whole exam** (parts 1 and 2, exercises 1 through 7).

Part 1. Only for those students who do the whole exam,
either because they did not partake in the first (midterm) test, or because they need (or wish) to improve on their midterm test grade.

$\boxed{5}$ 1. A university wishes to introduce a new educational system. During a one year test period, 10% of students are in a pilot project, which means they are taught according to the new system; all other students are taught according to the old system. After this year it turns out that 80% of the students in the pilot project were successful. Suppose that out of the total population, only 60% was successful. We hear about a particular student who was successful. What is the probability that this student was in the pilot project? Define appropriate events and use these to anwer the question.

$\boxed{2}$ 2. a. Give the probability definition of Laplace, and explain in which situations it can be applied.

$\boxed{2}$ b. Give the probability function, the expectation, and the variance of a gemetric random variable with parameter 0.2

3. Consider a deck of 52 cards, with 13 cards for each of the four suits (Spades/Hearts/Diamonds/Clubs). We draw four cards at random, without replacement.

$\boxed{3}$ a. What is the probability that precisely two cards are Hearts?

$\boxed{3}$ b. What is the probability that all four cards are in the same suit?

$\boxed{3}$ c. What is the expected number of suits that will be drawn?

Part 2. For all students.

4. Two random variables X and Y have a simultaneous probability function given by

$$P(X = i \text{ and } Y = j) = \frac{4}{13}(\frac{1}{2})^{i+j} \text{ for } j = 0, 1, 2, 3 \text{ and } i = 0, \cdots, 3 - j.$$

$\boxed{2}$ a. Determine the marginal probability function of X.
$\boxed{1}$ b. Are X and Y independent? (Motivate your answer)
$\boxed{2}$ c. Determine the probability function of $X + Y$.
$\boxed{2}$ d. Give the conditional probability function of X given that $Y = 0$.

5. The simultaneous probability density of X and Y is given by

$$f(x, y) = xy \qquad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2,$$

while $f(x, y) = 0$ elsewhere.
$\boxed{1}$ a. Determine the marginal density $f_X(x)$ for all values of x.
$\boxed{3}$ b. Compute $E(X)$ and $E(XY)$.
$\boxed{3}$ c. Determine $P(X > Y)$.

$\boxed{4}$ 6. The random variable X has a standard normal distribution. Determine the probability density function of X^2.

7. An amateur in electronics needs a resistor of at least 10 kΩ. Since he does not have one available, but happens to have a large number of 470 Ω resistors, he decides to take 22 of those, and put them in series. Assume the resistors are independent, with a uniformly distributed value between 420 en 520 Ω. It is important for the amateur that the total resistance R (being the sum of the 22 individual resistances) is not below 10 kΩ.
$\boxed{2}$ a. Determine the expectation and variance of the total resistance R.
$\boxed{2}$ b. Determine $\text{cov}(X_1, R)$, where X_1 is the resistance of one of the 22 individual resistors.
$\boxed{2}$ c. Approximate the probablity that the total resistance R is below 10 kΩ.
Before using them, the amateur checks each of his 470 Ω resistances, of which he has 35. The check is needed because he knows the 35 come from a large lot of resistors, of which 30 % is broken.
$\boxed{3}$ d. Approximate the probability that at least 22 of the 35 resistors are in order (use a continuïty correction).

Normering:

1	2		3			4				5			6	7				Totaal
	a	b	a	b	c	a	b	c	d	a	b	c		a	b	c	d	
5	2	2	3	3	3	2	1	2	2	1	3	3	4	2	2	2	3	45

$$\text{Second test grade (part 2)} = 1 + \frac{\text{number of points for exercises 4,5,6,7}}{3}$$

$$\text{Exam grade (parts 1 and 2)} = 1 + \frac{\text{number of points for exercises } 1-7}{5}$$

Final grade is determined as follows:

When you only do part 2:
Final grade = 2/5 * first test grade + 3/5 * second test grade, if both are ≥ 4.5.
Final grade = second test grade when this grade is < 4.5.

When you indicate to do the whole exam (or indicate nothing):
Final grade = exam grade.

Formulas

$$\text{var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{var}(X_i) + \sum_{i \neq j}\sum \text{cov}(X_i, X_j)$$

Hypergeometric distribution: $\text{var}(X) = n\frac{R}{N}\left(1 - \frac{R}{N}\right)\frac{N-n}{N-1}$

Geometric distribution: $\text{var}(X) = \frac{1-p}{p^2}$

Uniform distribution: $\text{var}(X) = \frac{1}{12}(b-a)^2$

The standard normal distribution

The table presents the cumulative distribution function Φ for a $N(0, 1)$-distributed random variable Z.

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx.$$

Example: $\Phi(1.65) = P(Z \leq 1.65) = 0.9505$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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Invullen in blokletters/To be completed by student

Cursusnaam/Coursename	Probability Theory	Datum/Date	Bladnr./Page no.
Cursuscode/Coursecode	191530062	31-10-2013	1
Studentnr./Student no. —	Voorl./Initials W.R.W.	Opleiding/Programme	Groepnr./Group no.
Naam/Name Scheinhardt		EE	—

Part 1

1) Sample space $S = \{\text{all students}\}$

Events: T: selected student is in the pilot project

A: " " is successful

Given: $P(T) = 0.1$

$P(A|T) = 0.8$

$P(A) = 0.6$

So $P(T|A) = \dfrac{P(T\cap A)}{P(A)} = \dfrac{P(A|T)\cdot P(T)}{P(A)} = \dfrac{0.8 \cdot 0.1}{0.6} = \dfrac{2}{15}$

2a) Event A (subset of sample set S) has probability

$P(A) = \dfrac{N(A)}{N(S)}$, where $N(A)$ is # outcomes in A and $N(S)$ is total # outcomes in S

Only valid if S is finite (so $N(S) < \infty$) and all outcomes are equally likely

b) Geometric r.v. X with parameter $p = 0.2$ has

probability function $P(X=n) = (1-p)^{n-1}\cdot p = 0.8^{n-1}\cdot 0.2$ $\quad n = 1, 2, \ldots$

expectation $EX = \frac{1}{p} = 5$

variance (see formulas) $Var(X) = \dfrac{1-p}{p^2} = \dfrac{0.8}{(0.2)^2} = 20$

3) a) Draw 4 from 52, of which 13 are hearts, without replacement

Hypergeometric: $P(2\text{ Hearts}) = \dfrac{\binom{13}{2}\binom{39}{2}}{\binom{52}{4}} = \dfrac{\frac{13\cdot 12}{2}\cdot\frac{39\cdot 38}{2}}{\frac{52\cdot 51\cdot 50\cdot 49}{4\cdot 3\cdot 2\cdot 1}} = 0.2135$

b) $P(\text{all 4 same suit}) = 4\cdot P(4\text{ Hearts}) = 4\cdot\dfrac{\binom{13}{4}\binom{39}{0}}{\binom{52}{4}} = 4\,\dfrac{13\cdot 12\cdot 11\cdot 10}{52\cdot 51\cdot 50\cdot 49}$

$P(4\text{ Spades}) = \ldots = P(4\text{ Hearts})$

$= 0.01056$

3c) Let $N = \#$ suits drawn. $\in \{1,2,3,4\}$.

$$EN = \sum_{n=1}^{4} n \cdot P(N=n)$$

$$P(N=1) = 0.01056 \qquad (\text{see b})$$

$$P(N=4) = \frac{\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{4}} = 0.1055$$

$$\underline{\underline{\text{or}}} = 1 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$$

$$P(N=3) = \underbrace{\frac{\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{0}}{\binom{52}{4}}}_{\substack{P(2 \text{ of suit } A \\ 1 \text{ of suit } B \\ 1 \text{ of suit } C \\ 0 \text{ of suit } D)}} \cdot \underbrace{4 \cdot 3}_{\substack{\#\text{ possible choices} \\ \text{for suit } A \text{ and } D \\ (\text{and hence } B \text{ and } C)}} = 0.0487 \cdot 12 = 0.5843$$

$$P(N=2) = 1 - P(N=1) - P(N=3) - P(N=4) = 0.2996$$

$$\underline{\underline{\text{or}}} = \underbrace{\frac{\binom{13}{1}\binom{13}{3}}{\binom{52}{4}} \cdot 4 \cdot 3}_{\substack{P(1 \text{ of one suit}, \\ 3 \text{ of another})}} + \underbrace{\frac{\binom{13}{2}\binom{13}{2}}{\binom{52}{4}} \cdot \binom{4}{2}}_{\substack{P(2 \text{ of one suit}, \\ 2 \text{ of another})}}$$

$$\text{So } EN = 0.01056 + 2 \cdot 0.2996 + 3 \cdot 0.5843 + 4 \cdot 0.1055$$

$$= 2{,}78$$

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Cursusnaam/Coursename: Probability Theory		Datum/Date: 31-10-2013	Bladnr./Page no.: 2
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Naam/Name: Scheinhardt			

~~Deel 2~~ Part 2

4) joint distr. in table:

$j \backslash i$	0	1	2	3	total
0	8/26	4/26	2/26	1/26	15/26
1	4/26	2/26	1/26	0	7/26
2	2/26	1/26	0	0	3/26
3	1/26	0	0	0	1/26
total	15/26	7/26	3/26	1/26	1

a) marginal distr. of X follows by adding columns, e.g. $P(X=0) = 8/26 + 4/26 + 2/26 + 1/26 = 15/26$

So:

i	0	1	2	3
$P(X=i)$	15/26	7/26	3/26	1/26

(check: sum = 1 ✓)

Or: $P(X=i) = \sum_{j=0}^{3} P(X=i, Y=j) = \sum_{j=0}^{3-i} \frac{4}{13}\cdot\left(\frac{1}{2}\right)^{i+j} = \frac{4}{13}\left(\frac{1}{2}\right)^i \sum_{j=0}^{3-i}\left(\frac{1}{2}\right)^j$

$= \frac{4}{13}\cdot\left(\frac{1}{2}\right)^i \cdot \frac{1-\left(\frac{1}{2}\right)^{3-i+1}}{1-\left(\frac{1}{2}\right)} = \frac{8}{13}\left(\left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^4\right)$

b) X and Y are indep. if $P(X=i, Y=j) = P(X=i)\cdot P(Y=j)$ for all i,j

Not true for ~~6~~ e.g. $i=j=2$: $P(X=2, Y=2) = 0 \neq \frac{3}{26}\cdot\frac{3}{26} = P(X=2)\cdot P(Y=2)$

So X and Y are not indep.

c) From table, e.g. $P(X+Y=1) = P(X=1, Y=0) + P(X=0, Y=1) = \frac{4}{26} + \frac{4}{26} = \frac{8}{26}$

Hence

k	0	1	2	3
$P(X+Y=k)$	8/26	8/26	6/26	4/26

(check: sum = 1 ✓)

or:

$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum_{i=0}^{k} \frac{4}{13}\left(\frac{1}{2}\right)^{i+(k-i)} = (k+1)\frac{4}{13}\cdot\left(\frac{1}{2}\right)^k$

d) $P(X=i \mid Y=0)$ for $i=0,\dots,3$

$= \frac{P(X=i, Y=0)}{P(Y=0)} = \frac{\frac{4}{13}\left(\frac{1}{2}\right)^{i+0}}{15/26} = \frac{8}{15}\left(\frac{1}{2}\right)^i$

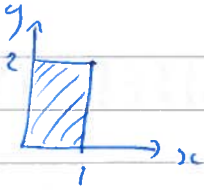
or from table, e.g. $P(X=0 \mid Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)} = \frac{8/26}{15/26} = \frac{8}{15}$

Hence

i	0	1	2	3
$P(X=i \mid Y=0)$	8/15	4/15	2/15	1/15

(check: sum = 1 ✓)

5) $f(x,y) = \begin{cases} xy & \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{else} \end{cases}$



a) $f_X(x) = 0$ for $x < 0$ and $x > 1$

for $0 \le x \le 1$: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 xy\,dy = \frac{1}{2}xy^2\Big|_0^2 = \frac{1}{2}x \cdot 4 = 2x$

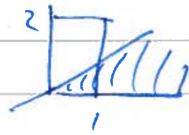
b) $EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

$EXY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dy\,dx = \int_0^1 \left(\int_0^2 x^2y^2\,dy\right)dx$

inner integral $\int_0^2 x^2y^2\,dy = \frac{1}{3}x^2y^3\Big|_{y=0}^2 = \frac{8}{3}x^2$

So $EXY = \int_0^1 \frac{8}{3}x^2\,dx = \frac{8}{9}x^3\Big|_0^1 = \frac{8}{9}$

c) integrate $f(x,y)$ over area where $x > y$



$P(X > Y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{x} f(x,y)\,dy\,dx$

$= \int_0^1 \left(\int_0^x xy\,dy\right)dx = \int_0^1 \left(\frac{1}{2}xy^2\Big|_0^x\right)dx = \int_0^1 \left(\frac{1}{2}x^3\right)dx$

$= \frac{1}{8}x^4\Big|_0^1 = \frac{1}{8}$

6) $X \sim N(0,1)$ let $Y = X^2$ then for $y \ge 0$

$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y}))$

$= 2\Phi(\sqrt{y}) - 1$

So using chain rule:

$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(2\Phi(\sqrt{y}) - 1\right) = 2\Phi'(\sqrt{y}) \cdot \frac{d}{dy}\sqrt{y}$

$= 2\varphi(\sqrt{y}) \cdot \frac{1}{2}y^{-\frac{1}{2}} = \frac{\varphi(\sqrt{y})}{\sqrt{y}} = \frac{1}{\sqrt{2\pi y}}e^{-\frac{1}{2}y}$, $y \ge 0$

also $f_Y(y) = 0$ for $y < 0$

7) a Let X_i, $i=1,\dots,22$ be the individual resistances

Then $X_i \sim U(420, 520)$ with $EX_i = 470$. Also $R = \sum_{i=1}^{22} X_i$

So $ER = E(\sum_{i=1}^{22} X_i) = \sum_{i=1}^{22} EX_i = 22 \cdot 470 = 10340$

also, since the X_i are independent, hence $\operatorname{cov}(X_i, X_j) = 0$ $(i \neq j)$ we have

$$\operatorname{Var} R = \operatorname{Var}(\sum_{i=1}^{22} X_i) = \sum_{i=1}^{22} \operatorname{Var}(X_i) = 22 \cdot \operatorname{Var} X_1 = 18333$$

formulas: $\operatorname{Var} X_1 = \frac{1}{12}(520-420)^2 = \frac{10000}{12} = 833{,}3$

b) $\operatorname{Cov}(X_1, R) = \operatorname{Cov}(X_1, X_1 + \dots + X_{22}) = \operatorname{Cov}(X_1, X_1) + \sum_{i=2}^{22} \operatorname{Cov}(X_1, X_i)$

$= \operatorname{Var}(X_1) + 0 = 833{,}3$

c) R is sum of 22 iid r.v.'s, so we can use the CLT.

$$P(R < 10000) = P\left(\frac{R - 10340}{\sqrt{18333}} < \frac{10000 - 10340}{\sqrt{18333}}\right)$$

$$\approx \Phi\left(-\frac{340}{135{,}4}\right) = \Phi(-2{,}51) = 1 - \Phi(2{,}51)$$

$$= 1 - 0{,}9940 = 0{,}006$$

d) Let X be # in order out of the 35. Then $X \sim \text{Binomial}(35, 0.7)$

So $EX = 35 \cdot 0.7 = 24.5$

$\operatorname{Var}(X) = 35 \cdot 0.7 \cdot 0.3 = 7.35$

now approximate X by $N(24.5, 7.35)$ using CLT:

$$P(X \geq 22) = P\left(\frac{X - EX}{\sqrt{\operatorname{Var}(X)}} \geq \frac{22 - 24\frac{1}{2}}{\sqrt{7.35}}\right)$$

$$\approx P\left(Z \geq \frac{21\frac{1}{2} - 24\frac{1}{2}}{\sqrt{7.35}}\right)$$ using cont. correction, $Z \sim N(0,1)$

$$= 1 - \Phi\left(-\frac{3}{\sqrt{7.35}}\right)$$

$$= \Phi(1.11) = 0{,}8665$$