Statistics & Probability (191506103)

Final Exam **(with 6 questions and 3 tables)**
Monday 29/10/18, 08:45 – 11:45

Full Marks : 35

1. Suppose X is an exponential(1) random variable, i.e., with probability density function (pdf) $f_X(x) = e^{-x}$, for $x > 0$. Define the new random variable $Y := \sqrt{X}$. Find the pdf of Y. [4]

2. Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers throughout the week. Because of limited supply the stock amount may not always be up to the full capacity of the tank and varies from week to week. Let X be the proportion of the total capacity that is stocked at the beginning of the week. Let Y be the proportion of the total capacity that is sold during the week. Obviously, $Y \leq X$. Suppose X and Y have the following joint probability density function:

$$f(x, y) = \begin{cases} 3x, & \text{if } 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a.) Find the (marginal) pdf of X. [2]

   (b.) What is the probability that in a given week more than half a tank will be sold given that three-quarter of the tank was stocked at the beginning? First determine the appropriate probability density function *completely.* [3]

   (c.) Note that $E(Y|X)$ can be interpreted here as the (conditional) expected amount of sale (as a proportion of the total capacity of the tank) during a week given the proportion that has been stocked at the beginning of that week. This would naturally be a function, $h(\cdot)$, of the stocked proportion X. Determine this function. [2]

3. Suppose a population can be modeled by a probability density function given by

$$f(x) = \frac{1}{2}(1 + \sqrt{\theta}\, x), \quad -1 \leq x \leq 1,$$

   where $0 < \theta < 1$ is an unknown parameter.

   (a.) Show that the mean of this population is given by $\mu = \frac{\sqrt{\theta}}{3}$. [1]

   (b.) Find the method of moment estimator for θ based on a random sample $X_1, X_2, \ldots, X_n$ from this population. [2]

   (c.) Is your estimator in part (b.) unbiased for θ? Explain. [2]

   [*Hint:* *You may use the facts that the population variance* $\sigma^2 = \frac{3-\theta}{9}$ *and for any random variable* Y, $E(Y^2) = Var(Y) + \big(E(Y)\big)^2$.
   If it helps, you may also consider $n = 3$ *for this part.*]

4. A portfolio consists of 48 stocks. The gain from each stock was supposed to be recorded in thousand of euros. For example, a gain of 1672 euro should have been recorded (in decimal) as 1.672. The summed up individual gains will then give the total gain (in thousand) from the portfolio.

   However, when summing up the gains from the 48 individual stocks, it is realized that all the recorded gains have been rounded off to the nearest integer; for example, above gain of 1.672 has been recorded as 2. In other words, the total gain calculated with these rounded-off data will be somewhat erroneous. Assume that the individual rounding-off errors are i. i. d. and have a (continuous) uniform distribution over the interval $(-\frac{1}{2}, \frac{1}{2})$.

   Compute, if needed approximately, the probability that the calculated (portfolio) gain will be within 2 (thousand) of the true gain (from the portfolio). [4]

   [Hint: Recall that if $X \sim Uniform(a, b)$, then $E(X) = \frac{b+a}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$.]

5. Consider an exponentially distributed population with unknown parameter θ (> 0), i.e., an observation, X, drawn randomly from this population has a pdf given by $f(x) = \theta\, e^{-\theta\, x}, \quad x > 0$.

   We want to do statistical inferences about the (unknown) parameter θ based on a random sample $X_1, X_2, \ldots X_n$ from this population. Now proceed as follows.

   (a.) Use the facts that the mgf of an Exponential(λ) random variable is given by $\lambda/(\lambda - t)$ and that for $\chi^2_{(n)}$ is $(1-2t)^{-n/2}$ to show that $2\theta X \sim \chi^2_{(2)}$. [2]

   (b.) Use part (a.) to show that $2n\theta\bar{X} \sim \chi^2_{(2n)}$. [Hint: Note that $2n\theta\bar{X} = \sum_{i=1}^{n}(2\theta X_i)$] [3]

   (c.) Use part (b.) to derive a formula for $100(1-\alpha)\%$ confidence interval for θ. [3]

6. A stock analyst claims to have devised a technique to select high quality stocks in the sense that it has high expected return with low volatility, i.e., a small standard deviation. One of the analyst's portfolios consisted of 24 stocks. The average return of the stocks (in the last period) was 11.50% and the standard deviation of the returns was 10.17%. The benchmark values for the type of stocks considered were: a mean of 10.10% and a standard deviation of 15.67%. Assume that the returns of the stocks in consideration are normally distributed and independent.

   (a.) Test at 0.05 level of significance whether it can be concluded that the expected return from a stock selected by the analyst indeed beats the benchmark return of 10.10%. [5]

   (b.) While performing the test in part (a.) you have made some model assumptions. Can you think of other possible (and not too wild) model assumptions? Briefly describe how your test procedure would have changed under those model assumptions. [You do not actually need to perform the test.] [2]

Final grade $= \left\{ \frac{\text{score on exam}}{35} \times 9 + 1 \right\}$ (rounded off to an integer)