Statistics & Probability (191506103)

Final Exam **(with 7 questions and 3 tables)**
Monday 30/10/17, 13:45 – 16:45

Full Marks : 35

1. Suppose X is an exponential(1) random variable, i.e., with probability density function (pdf) $f_X(x) = e^{-x}$, for $x > 0$. Define the new random variable $Y = \frac{1}{X+1}$. Find the pdf of Y. [4]

2. Suppose X and Y have the joint probability density function (pdf) given by

$$f(x,y) = e^{-x}, \quad \text{for } 0 \le y \le x, \qquad \text{and} \quad = \; 0, \quad \text{elsewhere.}$$

(a.) Find the (marginal) pdf of X. [2]

(b.) Find the conditional pdf, $f_{Y|X=x}(\cdot)$, of Y given $X = x \quad (x > 0)$. [1]

(c.) Recall that $E(Y|X)$ is a random variable, which is a function of X. Express the random variable in terms of X. [2]

3. Suppose a door-to-door salesman has 10% chance of selling his products to a randomly selected person. The salesman is not very ambitious in his job. Everyday he stops working whenever he makes his first sale. Furthermore, assume that the sales patterns are independent from house to house and from day to day.

(a.) Show that the probability that on a given day the salesman will need to visit at least 25 houses before he stops working is 0.0798. [2]

$$\left[\textit{Hint: You may use the fact that } \sum_{x=k}^{\infty} a^x \equiv a^k + a^{k+1} + \cdots = \frac{a^k}{1-a}, \quad \textit{for } \; 0 < a < 1. \right]$$

(b.) Suppose that the salesman works 220 days a year (out of the 52 weeks, he takes 8 weeks off/vacation). Calculate the probability that he will need to visit at least 25 houses in at most 25 days of a given year. Give as accurate an answer as you can from the materials provided. [4]

[Answers obtained directly from an advanced/graphical calculator would not be accepted.]

4. Suppose a random sample of size n is taken from a geometric population with parameter p $(0 < p < 1)$, with the probability mass function

$$P(X = x) = (1-p)^{x-1}p, \quad x = 0, 1, 2, 3, \ldots.$$

Find the maximum likelihood estimator of p. [4]

5. Recall that the moment generating function (mgf) of a $Binomial(n, p)$ random variable is given by $(1-p+pe^t)^n$.

   (a.) Suppose X and Y are independent with $X \sim Binomial(m, p)$ and $Y \sim Binomial(n, p)$. Let $W = X + Y$. Identify the probability distribution of W by first calculating the moment generating function (mgf) of W. [2]

   (b.) Let V be a random variable with mgf: $m_V(t) = \left[\frac{1}{4}(3+e^t)\right]^5$. Find $P(V < 2)$. [2]

6. A stock analyst claims to have devised a mathematical technique for selecting high quality stocks. A collection of certain number of stocks is also called a mutual fund. The analyst promises that his mutual fund will have a high 10-year annualized return with low volatility. After 10 years, one of the analyst's funds/portfolios is examined. The portfolio/fund consisted of 24 stocks. The performance of the stocks included in the portfolio showed an average 10-year annualized return of 11.50% and a standard deviation of 10.17%. The 10-year annualized return on other similar stocks was approximately 10.10%. Assume that the 10-year annualized returns of the stocks in consideration are normally distributed.

   (a.) Obtain a 95% confidence interval of the expected 10-year annualized return for the stocks selected by the analyst. [3]

   (b.) Explain how the calculation of your confidence interval in part (a.) would have changed if the true variability of the performance of the stocks were known. [2]
   [Hint: You do not need to actually calculate any new interval.]

7. Pond's Age-Defying Complex, a cream with alpha-hydroxy acid, advertises that it can reduce wrinkles and improve the skin. In a study published in Archives of Dermatology (June 1996), 33 middle-aged women used a cream with alpha-hydroxy acid for twenty-two weeks. At the end of the study period, a dermatologist judged whether each woman exhibited skin improvement. The study resulted in 24 "I"s and 9 "N"s, (where I = improved skin and N = no improvement).

   (a.) Do the data provide sufficient evidence to conclude that the cream will improve the skin of more than 60% of middle-aged women? Test using $\alpha = .05$. [5]

   (b.) Find and interpret the P-value of the test. [2]

$$\text{Final grade} = \left\{\frac{\text{score on exam}}{35} \times 9 + 1\right\} \text{ (rounded off to an integer)}$$