Robust Control — EXAM

Course code:	191560671
Date:	18-04-2017
Time:	13:45–16:45 (till 17:30 for students with special rights)
Course coordinator & instructor:	G. Meinsma
Type of test:	open book
Allowed aids during the test:	printed lecture notes, basic calculator

---

1. Consider the non-rational transfer function

$$G(s) = \frac{1}{1 + \frac{1}{2}\mathrm{e}^{-s}}$$

defined for s in the ORHP.

   (a) Show that $G(s)$ is well defined for every s in the ORHP and that $G \in \mathbb{H}_\infty$.

   (b) Determine $\|G\|_{\mathbb{H}_\infty}$.

2. (Assume all systems are SISO.) Not infrequently disturbances w enter the plant at the input (see w in the second figure on page 25 of the notes). Suppose that the system is internally stable and that the loop gain has integrating action (i.e. $L(s)$ has a pole at $s = 0$). Prove that the DC-gain of $H_{y/w}(s)$ is zero if and only if the *controller* has integrating action.

3. Chapter 4 introduces the *gain margin* k_m, *phase margin* ϕ_m and *modulus margin* s_m.

   (a) Show that $0 < s_\mathrm{m} < 1$ implies a guaranteed gain margin of at least $1/(1 - s_\mathrm{m})$.

   (b) Does $k_\mathrm{m} > 1$ imply a guaranteed $s_\mathrm{m} > 0$?

   (c) Does $\phi_\mathrm{m} > 0$ imply a guaranteed $s_\mathrm{m} > 0$?

4. In § 8.1 we designed a stabilizing controller for the plant $P(s) = 1/s^2$. Unfortunately all sensitivity functions S, T designed in § 8.1 appear to have peaks with $\|S\|_{\mathbb{H}_\infty} > 1$ and $\|T\|_{\mathbb{H}_\infty} > 1$.

   (a) For this plant is there a proper stabilizing $K(s)$ that achieves $\|S\|_{\mathbb{H}_\infty} \leq 1$?

   (b) For this plant is there a proper stabilizing $K(s)$ that achieves $\|T\|_{\mathbb{H}_\infty} \leq 1$?

   Explain your answers.

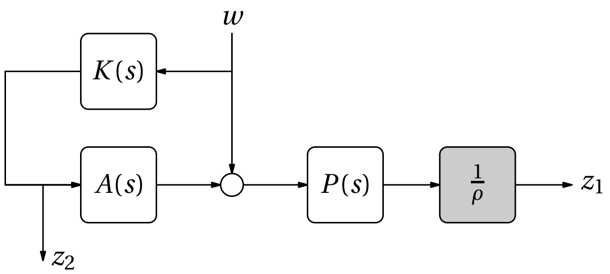
5. *Chapter 6:* Consider the system

$$\dot{x} = \frac{1}{4}x + u, \qquad x(0) = x_0.$$

with cost $\int_0^\infty x^2(t) + (x(t)+u(t))^2 \, \mathrm{d}t$. Determine the solution P of the corresponding LQ-Riccati equation and determine the LQ-optimal state feedback $u = -Fx$.

6. Are all polynomials in the family of polynomials $[1,2]s^3 + [1,2]s^2 + [1,2]s + [1,2]$ stable?

7. *Disturbance feedforward.* Sometimes we can measure a disturbance w that acts on a plant $P(s)$. It then makes sense to try to counter-act this disturbance. Hagander and Bernhardsson suggested the following scheme:



in which $\rho > 0$ is a tuning parameter and $A(s)$ is some given actuator system, and then they suggest to minimize $\|H_{z/w}\|_{\mathbb{H}_\infty}$ over all stabilizing $K(s)$.

   (a) Under what conditions on $A(s), P(s), K(s)$ is this system internally stable?

   (b) Formulate this is a standard $\mathbb{H}_\infty$ problem (that is, determine the generalized plant $G(s)$).

   (c) Suppose that $P(s) = A(s) = 1/(s+1)$. What can you say about the order of the $\mathbb{H}_\infty$-optimal controller K? (For proper rational $K(s)$ the "order" is the degree of its denominator polynomial.)

8. Table 9.1 of the lecture notes claims that the interconnection matrix for $P = P_0(I + V\Delta W)^{-1}$ is $H = -W(I + KP_0)^{-1}V$. Verify this result. Your derivation must hold for MIMO systems as well.

problem:	1	2	3	4	5	6	7	8
points:	2+2	3	2+2+2	2+3	4	3	2+2+3	4

Grade: $= 1 + 9\frac{p}{p_{\max}}$ (possibly with homework correction of ≤ 0.6)

TELEGRAM STYLE ANSWERS:

1. (a) If $\operatorname{Re}(s) > 0$ then $|e^{-s}| < 1$ so then $1 + \frac{1}{2}e^{-s}$ is nonzero. Hence $G(s)$ is well defined for all $\operatorname{Re}(s) > 0$ and (obviously) differentiable. Also, since $|1 + \frac{1}{2}e^{-s}|$ is at least $1/2$ on ORHP we have $|G(s)| \leq 2$ for all s in ORHP. So $G(s)$ is defined, differentiable and bounded on ORHP. Hence it is in $\mathbb{H}_\infty$.

   (b) $|G(i\omega)|$ is maximal iff $|1 + \frac{1}{2}e^{-i\omega}| = 1/2$ which is the case iff $\omega = \pi + 2k\pi$. So $\|G\|_{\mathbb{L}_\infty} = 2$ (and this equals the $\|G\|_{\mathbb{H}_\infty}$ norm.)

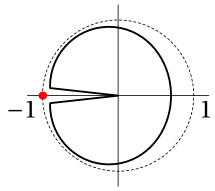
2. $H_{y/w} = P/(1+PK)$ and because of internal stability its DC-gain exists. K is not the zero function because L has integrating action, so we can express $H_{y/w}$ as $H_{y/w} = T/K$. Because of the integrator in L we have $T(0) = 1$ and therefore the DC-gain of $H_{y/w}$ equals $1/K(0)$. This is zero iff $K(s)$ has one or more poles at $s = 0$.

   Another proof (using polynomials): let $P = N/D, K = B/A$. Then $H_{y/w} = P/(1+PK) = NA/\chi_{\text{cl}}$ so DC gain is zero iff $N(0) = 0$ and/or $A(0) = 0$. Now $N(0) = 0$ is impossible as that would mean $P(0) = 0$ which contradicts that L has integrating action and internal stability. So DC gain is zero iff $A(0) = 0$, i.e. iff $K(s) := B(s)/A(s)$ has a pole at $s = 0$.

3. (a) By definition, the ball with radius s_m around -1 contains no $L(i\omega)$. Hence when $L(i\omega)$ is real and negative then either $L(i\omega) > -(1 - s_m)$ or $L(i\omega) < -(1 + s_m)$. In the first case the (upward) gain margin $1/L(i\omega)$ is at least $1/(1 - s_m)$ and in the second case the (downward) gain margin is at most $1/(1 + s_m)$... If only upward gain margins count then the claim is valid

   (b) No: because the "packman-shaped" Nyquist plot (see below) has infinite gain and phase margin, yet s_m is practically zero.

   (c) No (same argument)



4. (a) No because of Thm 5.3.1: Since K is proper and $P(s) = 1/s^2$, the loop gain $L = PK$ satisfies the conditions of Bode's Sensitivity Integral (Thm 5.3.1) so we have $\int_0^\infty \ln(|S|)\,d\omega = 0$ (or if L has poles in ORHP: $\int_0^\infty \ln(|S|)\,d\omega = \pi \sum \operatorname{Re}(p_i) > 0$, see Eqn (5.7)). Conclusion: $\ln|S|$ *is zero or postive on average.* Since $\ln(|S|) = -\infty$ (by the fact that L has an integrator) Bode's sensitivity integral implies we have $\ln|S| > 0$ for certain other frequencies, that is $|S| > 1$. Hence $\|S\|_\infty > 1$.

   (b) No because of Lemma 5.3.3: Since our L has a double integrator (because P has and K stabilizes) it satisfies the conditions of Lemma 5.3.3. Since $\ln|T(\infty)| = -\infty$ (because $L = PK$ is strictly proper) it must be that $|T| > 1$ for certain frequencies. Hence $\|T\|_{\mathbb{H}_\infty} > 1$.

5. So $A = 1/4, B = 1$ and

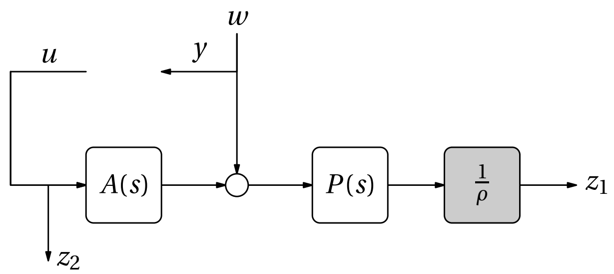
$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{D} u.$$

Thm 6.1.5 says: $F = P+1$ and the Riccati equation then is $\frac{1}{4}P + P\frac{1}{4} + 2 - (P+1)^2 = -(P^2 + \frac{3}{2}P - 1) = -(P+2)(P-1/2)$. So $F = P+1$ is either -1 or 1.5. The first ($F = -1$) is not stabilizing. The other ($F = 1.5$) is stabilizing, so: $F = 1.5$ and $P = 1/2$ (and then $A - BF = -1.25$).

6. According to Kharitonov we need only consider four specific ones. The Kharitonov polynomial $s^3 + s^2 + s + 1 = (s+1)(s^2+1)$ is not stable (which can be checked with Routh-Hurwitz (do it yourself!))

7. (a) There is no feedback, so all three need to be stable.....

   (b) It is good to add some signal labels and to remove $K(s)$:

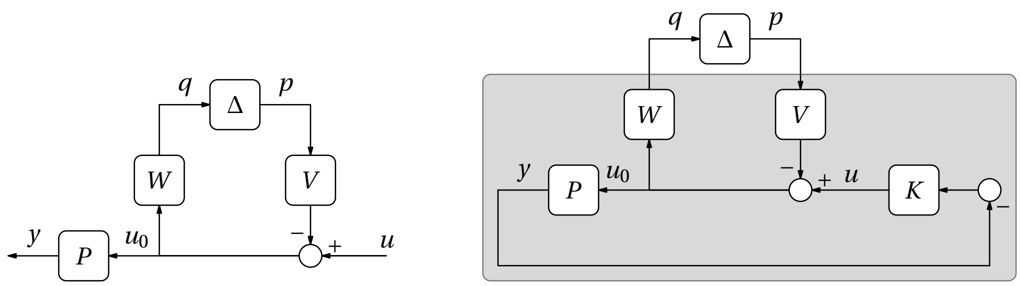


   This shows that

$$G = \begin{bmatrix} \frac{1}{\rho}P & \frac{1}{\rho}PA \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   (c) The state dimension of G is 2 so the "suboptimal" controller K (according to Thm. 8.2.2) has order 2. (Actually often (but not always) the *true optimal* controller has reduced order, so order 1 in this case. This happens "a lot" but not always.)

8. Claim: the configuration of $P_{y/u} := P_0(I + V\Delta W)^{-1}$ is as shown on the left:



Check: from the configuration (on the left) it follows that $u_0 = -Vp + u$ and $p = \Delta W u_0$, so $u_0 = -V\Delta W u_0 + u$, that is,

$$(I + V\Delta W)u_0 = u$$

and, therefore, $y = P_0 u_0 = P_0(I + V\Delta W)^{-1}u$. Correct.

Now add controller and separate Δ (shown in above figure(right)). Discarding Δ, the interconnection matrix $H_{q/p}$ follows from the configuration: $u_0 = -Vp + u = -Vp - KP_0u_0$. Hence $(I + KP_0)u_0 = -Vp$ and so $u_0 = -(I + KP_0)^{-1}Vp$. Since $q = Wu_0$ we get what we had to prove: $q = (-W(I + KP_0)^{-1}V)p$.