Random Signals and Filtering (201200135)

Faculty of EEMCS, University of Twente

Sample Exam (with 4 questions) | Full Marks: 30

Weekday, dd/mm/yyyy, hh:mm – hh:mm | Instructor: P. K. Mandal

This is a closed book exam. Formulate your answers clearly, with proper motivation.
Present your answers in a well-structured manner.

1. Suppose $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is a function.

   a) Consider a function $\mathcal{P}$ defined on subsets of Ω satisfying:

   $$\mathcal{P}(\ \{n,\, n+1,\, \ldots,\, m\}\ ) = f(m) - f(n-1), \quad \text{for } \ m \geq n \geq 1.$$

   What characteristics must the function f necessarily have in order for $\mathcal{P}$ to be able to satisfy the axioms of a probability measure? [3]

   b) If $\mathcal{P}$ were to satisfy

   $$\mathcal{P}(\ \{n,\, n+1,\, \ldots,\, m\}\ ) = f(m) - f(n), \quad \text{for } \ m \geq n \geq 1,$$

   how would your answer to question a) change? [2]

2. Consider the following linear non-Gaussian system: for $k \geq 0$,

   $$X_{k+1} = -X_k + W_k \qquad \text{and} \qquad Y_k = X_k + V_k,$$

   where the initial state X_0 and the noises W_k, V_k $(k \geq 0)$ are all Unif$(0, 1)$ random variables. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

   a) Show that the pdf of Y_0 is given by $\quad p_{Y_0}(y_0) = \begin{cases} y_0 & 0 < y_0 \leq 1 \\ 2 - y_0 & 1 \leq y_0 < 2. \end{cases}$ [2]

   b) Show that the posterior distribution of X_0 (conditional on $Y_0 = y_0$) is best described as $X_0|Y_0{=}y_0 \sim \text{Unif}\,(0, y_0)$, if $0 < y_0 \leq 1$, $\ \text{Unif}\,(y_0 - 1, 1)$, if $1 \leq y_0 < 2$, and for other values of y_0, it is undefined.
   In other words, show that [2]

   $$p_{X_0|Y_0=y_0}(x_0) = \begin{cases} \text{undefined,} & \text{if } y_0 \notin (0, 2) \\ \frac{1}{y_0}, & \text{if } 0 < y_0 \leq 1 \ \text{ and } \ 0 \leq x_0 \leq y_0, \\ \frac{1}{2 - y_0}, & \text{if } 1 \leq y_0 < 2 \ \text{ and } \ y_0 - 1 \leq x_0 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   c) Calculate, explicitly, the estimate $\hat{X}_0 = E(X_0|Y_0)$ and subsequently, the predicted estimate $E[X_1|Y_0]$ in terms of $\hat{X}_0$. [3]
   [You may use the fact that if $U \sim \text{Unif}(a, b)$, then $E(U) = \frac{a+b}{2}$.]

3. Recall that $E(X \mid Y)$ can be thought of as the projection of X onto $\mathcal{V}$, the space of square integrable (Borel) functions of Y. On the other hand, $E_{\text{aff}}(X \mid Y)$ is the projection onto $\mathcal{V}_{\text{aff}}$, the space of linear function(al)s of Y, and is given by

$$E_{\text{aff}}(X \mid Y) = E(X) + \operatorname{Cov}(X, Y) \operatorname{Cov}(Y)^{-1} (Y - E(Y)).$$

In general, $E(X \mid Y)$ and $E_{\text{aff}}(X \mid Y)$ need not be the same. However, if (X, Y) is jointly Gaussian, then they are same. Through the following steps you will prove this result.

a) Use the characterization properties of a projection to show the following [2]

$$E(X) = E[\, E_{\text{aff}}(X|Y)\,] \quad \text{and} \quad E(XY) = E[\, E_{\text{aff}}(X|Y)\, Y\,].$$

For the following parts, suppose that (X, Y) is jointly Gaussian. Also, you may use any property of joint Gaussianity and independence, without proof, after stating it clearly.

b) Show that $e := X - E_{\text{aff}}(X|Y)$ is independent of Y. [3]

[Thus, e is indepenent of any (Borel) function $g(Y)$, as well.]

c) Argue that $E(X \mid Y) = E_{\text{aff}}(X \mid Y)$. [2]

4. Consider the following nonlinear system. For $k \geq 0$,

$$\begin{aligned} X_{k+1} &= \sqrt{X_k}\, W_k \\ Y_k &= X_k \sqrt{V_k}, \end{aligned}$$

where the initial state X_0 and the noises W_k, V_k $(k \geq 0)$ are all Uniform$(0, 1)$. Furthermore, X_0 and the sequences $\{W_k\}$ and $\{V_k\}$ are mutually independent.

a) Suppose we want to implement a particle filter (PF) to this system with the importance density $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1})$. Describe the required recursion in the form of an algorithm/pseudo-code. [4]

More preceisely, given $\{(x^i_{k-1}, w^i_{k-1}),\ i = 1, 2, \ldots, N\}$, the weighted particle representation of the posterior at time $(k-1)$ and the new measurement y_k at time k, describe how to obtain the particle representation, $\{(x^i_k, w^i_k),\ i = 1, 2, \ldots, N\}$, of the posterior at time k.

Assume that you have access to a command `rand(a,b)` to generate a sample from Unif(a, b) distribution. Everything else in your code should be self explanatory, i.e., in terms of standard arithmetic operations and known structures like `if` and `for` loop.

b) How can you extract the posterior mean $E(X_k \mid Y_{0:k})$ and the variance $\operatorname{Var}(X_k \mid Y_{0:k})$ from the particle representation? [1]

c) Often, a resampling step is performed in a PF algorithm. Discuss, briefly, the problem that the resampling step tries to overcome. [2]

d) Often, one talks about the *optimal* importance/proposal density in a PF. What is it? In what sense is it optimal? [2]

e) Could we have used (extended) Kalman filter for this system to extract the posterior information about x_k? If yes, how would you proceed? If not, what would you have needed to be able to apply EKF? [2]