Online Exam: Random Signals and Filtering (201200135)
Thursday, 09/07/2020, 08:45 – 11:15

- You are allowed to use a calculator, the two reference books/notes (by Hajek and Stoorvogel), the notes provided by me and those you took during the classes.
- All answers must be motivated; arguments and proofs must be complete. You may use any result you like without proof, unless asked to prove explicitly. You should aways provide references to te used results. References must be to the textbook(s) and/or lecture slides.
- At the end of this written exam, there will be a supplementary oral exam for a group of randomly selected students. The oral will last maximum 15 minutes per student. The oral will be recorded on video and will only be used for quality control (by, e.g., the examination board, accreditation committee and the likes).
- You have the right to refuse the oral exam being recorded. In that case, however, you may not participate in this online exam. You have to wait till the situation (around Corona-virus) normalizes and another on-campus exam can be scheduled.

The written test is worth 25 points.

Your grade on the written exam will be: $1 + \frac{\text{obtained\_points}}{25} \times 9.$

Please read carefully: *By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test. If significant irregularities are detected, the examiner/examination board may declare the test result of an individual or those for all participants invalid.*

In order for the test to be graded, the following text must be copied (handwritten and signed) on the first page of your solution, along with a photocopy of your student-id:

> **"I have made this test to the best of my own ability, without seeking or accepting help of any source not explicitly allowed by the conditions of the test. Neither have I provided help to anybody else.**
> **Also, I give my consent to the oral exam being recorded on video for the purpose of quality control."**
> **[Name, Student no., Location, Date, Signature]**

Questions start on the next page.

[There are 5 questions (with 9 parts in total) Full Marks : 25]

1. The (orthogonal) projection theory, in particular, onto the space of linear functionals of random variables, provides a neat way of deriving the Kalman filter recursion equations. In this question, you will derive the relevant formulas for the projection onto the space of quadratic functions of a random variable.

   Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Suppose we are interested in the (orthogonal) projection of X onto the space of quadratic functions of Y, given by

   $$\mathcal{V} := \{a + b\,Y + c\,Y^2 : a, b, c \in \mathbb{R}\}.$$

   Suppose all the moments/expectations of type $E(X^i\,Y^j)$ exist for all $i, j = 0, 1, 2, \ldots$; denote them by $\mu_{i,j}$. For example, $\mu_{1,1} = E(XY)$ and $\mu_{0,2} = E(Y^2)$.

   Find the (normal) equations for the coefficients a^*, b^*, c^*, in terms of $\mu_{i,j}$'s, so that $a^* + b^*\,Y + c^*\,Y^2$ becomes the projection, $P_{\mathcal{V}}X$, of X onto $\mathcal{V}$. [5]

2. Suppose the initial state, X_0, of a linear system is Gaussian with mean 10 and variance 5. The state and measurement at time $t = 1$ are given by:

   $$X_1 \;=\; 0.8\,X_0 + W_0 \quad \text{and} \quad Y_1 \;=\; 2\,X_1 + V_1,$$

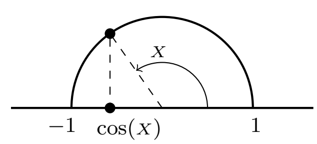
   where the state-noise W_0 and the measurement-noise V_1 are also Gaussian with means zero and variances 3 and 1, respectively. Assume that X_0, W_0 and V_1 are independent.

   In the following you will derive the Kalman filter at time $t = 1$, i.e., the (conditional) probability distribution of state X_1 given the measurement $Y_1 = y_1$.

   a) Find the probability distribution of the vector (X_1, Y_1), by providing proper references to the results you are using. [2]

   b) Find the conditional probability distribution of X_1 given $Y_1 = y_1$. [2]

3. Suppose an object is moving on the semi-circle of radius 1, where the position of the object is identified by its angle X to the horizontal line, as shown on the right. We can observe only its projection onto the horizontal axis, i.e., $\cos(X)$, and that too mixed with an additive noise. See figure on the right.

   The current position can be anywhere on the semi-circle, i.e., $X \sim \text{Unif}(0, \pi)$. Consequently, the current observation is $Y := \cos(X) + V$, where V is the (additive) observation noise. Assume that V is distributed as $V \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2})$, and independent of X.

   In the following you will obtain the probability density function of current observation Y.

   a) First obtain the conditional pdf $f_{Y|X=x}(y)$ and then express the joint density $f_{X,Y}(x, y)$ as compactly as possible. [2]

   b) Marginalize the joint density to obtain the pdf of Y. [3]

   [Hint: It may help to consider three regions $y < -\frac{1}{2}$, $y > \frac{1}{2}$ and the rest, separately.]

4. Consider the following nonlinear system: for $k \geq 0$,

$$X_{k+1} = \sqrt{X_k}\, W_k \qquad \text{and} \qquad Y_k = X_k \sqrt{V_k},$$

where the initial state X_0 and the noises W_k, V_k $(k \geq 0)$ are all Uniform$(0, 1)$. Furthermore, X_0 and the sequences $\{W_k\}$ and $\{V_k\}$ are mutually independent.

a) Suppose we want to implement the standard particle filter (PF) to this system, i.e., with the importance density to be the transition density: $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1})$.
Give the pseudo code for a generic iteration step of the particle filter. [4]

More precisely, describe how you will use $\{(x_{k-1}^i, w_{k-1}^i),\ i = 1, 2, \ldots, N\}$, the weighted particle representation of the posterior at time $(k-1)$, and the measurement y_k at current time k, to obtain the particle representation of the current posterior: $\{(x_k^i, w_k^i),\ i = 1, 2, \ldots, N\}$.

The code should be self explanatory and readily implementable, i.e., in terms of known/standard (mathematical) functions and known structures like `if` and `for` loop. You may assume that you have access to the following programming-commands.

`rand(a,b)` that generates a sample from a Unif(a, b) distribution.

`NormPDF(x,m,s)` that evaluates the $\mathcal{N}(\mathtt{m}, \mathtt{s}^2)$-density at the point `x`

`RandNorm(m,s)` that generates a sample from a $\mathcal{N}(\mathtt{m}, \mathtt{s}^2)$-r.v.

`RandPMF(x_vec, p_vec, n)` that produces a random sample of size "`n`" from the discrete distribution with values "`x_vec`" and corresponding probabilities "`p_vec`".

b) How will you obtain the posterior variance $\text{Var}(X_k \mid Y_{0:k})$ from the weighted particles? [1]

c) Often one talks about the *optimal* importance/proposal density in a PF. What is it? In what sense is it optimal? [2]

5. In the derivation of weight update equation corresponding to the standard particle filter, one requires the following equality:

$$\frac{p(x_{0:k}^i|y_{0:k})}{p(x_{0:k}^i|y_{0:k-1})} = \frac{p(y_k|x_k^i)}{p(y_k|y_{0:k-1})}. \qquad (\star)$$

The underlying assumption is, of course, that $\{x_k, y_k\}$, constitute a standard 1st order (partially observed) state space model of the form

$$\text{(state)} \quad x_k = f_{k-1}(x_{k-1}, u_{k-1}), \qquad \text{(measurement)} \quad y_k = h_k(x_k, v_k),$$

where the noises $\{u_k\}$ and $\{v_k\}$ are independent.

Show that the equality $(\star)$ indeed holds. [4]