Random Signals and Filtering (201200135)

Final Exam (with 4 questions) Full Marks: 30
Thursday 19/04/2018, 13:45 – 16:45 Instructor: P. K. Mandal

Formulate your answers clearly and present them in a well-structured manner.

1.(a.) Let Ω be a set. Give the definition of a field and a sigma-field on Ω. [3]

(b.) Consider the collection $\mathbb{A}$ of semi-open intervals of $(0,1]$ and the finite unions of disjoint semi-open intervals, i.e., $(a,b],\ 0 \le a \le b \le 1$.

$$\mathbb{A} = \left\{ \bigcup_{i=1}^{n} (a_i, b_i], \quad n \ge 1, \quad (a_i, b_i] \cap (a_j, b_j] = \emptyset, \ \forall\, i \ne j = 1, 2, \ldots, n \right\}.$$

It is known that $\mathbb{A}$ is a field on $(0,1]$. The proof hinges on the following two simplified properties. Show these two properties: [2]

(i.) Let $A = (a_1, b_1] \cup (a_2, b_2]$, union of two disjoint semi-open intervals of $(0,1]$. Then A^c can be represented as a finite union of disjoint semi-open intervals of $(0,1]$.

(ii.) Let $A, B \in \mathbb{A}$ be disjoint. Then $A \cup B$ is also in $\mathbb{A}$.

2. Consider the following linear non-Gaussian system: for $k \ge 0$,

$$X_{k+1} = -X_k + W_k \qquad \text{and} \qquad Y_k = X_k + V_k,$$

where the initial state X_0 and the noises W_k, V_k $(k \ge 0)$ are all Unif(0, 1) random variables. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

(a.) Show that the pdf of Y_0 is given by $p_{Y_0}(y_0) = \begin{cases} y_0 & 0 < y_0 \le 1 \\ 2 - y_0 & 1 \le y_0 < 2. \end{cases}$ [3]

(b.) Show that the posterior distribution of X_0 (conditional on $Y_0 = y_0$) is given by [2]

$$p_{X_0|Y_0=y_0}(x_0) = \begin{cases} \frac{1}{y_0}, & \text{for } 0 \le x_0 \le y_0, \quad \text{i.e., Unif}(0, y_0), \quad \text{if } 0 < y_0 \le 1 \\ \frac{1}{2-y_0}, & \text{for } y_0 - 1 \le x_0 \le 1, \quad \text{i.e., Unif}(y_0 - 1, 1), \quad \text{if } 1 \le y_0 < 2 \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Provide, subsequently, the (point) estimate $\hat{X}_0 = E(X_0 \mid Y_0)$. [1]

(c.) Use the *law of iterated expectation*: $E[X|Y] = E\Big[E[X|Y,Z] \Big| Y \Big]$ to obtain the predicted estimate $E[X_1|Y_0]$ in terms of $\hat{X}_0$. [2]

3. Recall that $E_{\text{aff}}(X \mid Y)$ is the "best" predictor of X as a linear/affine function of Y, of the form $a + bY$. More precisely,

$$E_{\text{aff}}(X \mid Y) = E(X) + \text{Cov}(X,Y)\,\text{Cov}(Y)^{-1}\,(Y - E(Y)).$$

In a linear-Gaussian dynamical system of (unobserved) signals (X_k) and measurements (Y_k), the posterior mean $E(X_k|Y_{0:k})$ can be expressed as a linar/affine combination of $Y_0, Y_1, \ldots, Y_k$. Kalman filter (KF) exploits this fact. In the calculation, KF uses the concept of linear innovations.

(a.) Explain what linear innovations are corresponding to a sequence of measurements $Y_0, Y_1, \ldots, Y_n$. Also, explain, briefly, how they help in the derivation of KF. [3]

(b.) Consider the following nonlinear system with additive Gaussian noises: for $k \geq 0$,

$$X_{k+1} = X_k + W_k \quad \text{and} \quad Y_k = X_k^2 + V_k, \qquad (\star)$$

where the initial state X_0, and for $k \geq 0$, the noises W_k, V_k are all $N(0,1)$. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

This system is not suitable for applying KF. However the concept of linear innovation can still be applied. Obtain the first two innovations. [3]

[Hint: Use different properties of covariances. You may also use the fact that for $Z \sim N(0,1)$, $E(Z^2) = \text{Var}(Z) = 1$ and $\text{Var}(Z^2) = 2$.]

4. Consider again the (real-valued) nonlinear system $(\star)$ in question 3(b). It is not suitable for KF, because KF requires the measurement model to be of the form $Y_k = H_k X_k + V_k$. However, one may apply the extended Kalman filter (EKF) or a particle filter (PF).

(a.) Suppose we would like to implement an EKF to the system $(\star)$. Explain how you would transform, given the estimates $\hat{x}_{k-1|k-1}$ and $P_{k-1|k-1}$ of time $(k-1)$, the observation equation, in order to implement EKF. Give explicit formulas. [3]

(b.) Give the pseudo-code for the generic iteration step of a standard PF applied to the system $(\star)$, i.e., how to obtain $(x_k^i, w_k^i)_{i=1}^N$ from $(x_{k-1}^i, w_{k-1}^i)_{i=1}^N$ and the measurement y_k. [5]

Assume that you have access to a command `randn` to generate a sample from $N(0,1)$ distribution, and a command `resample(x_vec, p_vec, n)` to draw a random sample of size "n" from the discrete distribution with values "x_vec" and corresponding probabilities "p_vec". Everything else in your code should be self explanatory, i.e., in terms of known function or known structures like `if` and `for` loop.

(c.) Suppose one is interested in the (posterior) probability of the signal exceeding 10, i.e., $P(X_k > 10 \mid Y_{0:k})$. How would you estimate it from the weighted particles? [1]

(d.) In the context of a PF, one often talks about the *optimal* importance/proposal density. What is it and how does it help? [2]