Ewout.

Optimal Control
(course code: 191561620)

Date:	22-01-2020
Place:	Sports Centre, SC3
Time:	08:45–11:45 (till 12:30 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

1. Consider the nonlinear system

$$\dot{x}_1(t) = x_1(t)x_2(t) + x_2(t)$$
$$\dot{x}_2(t) = -(1 + x_2^2(t))x_1(t) - x_2(t)$$

   (a) Determine all equilibria $\bar{x} \in \mathbb{R}^2$.

   (b) Linearize the system around equilibrium $\bar{x} = (0,0)$

   (c) Is $\bar{x} = (0,0)$ an asymptotically stable equilibrium of the nonlinear system?

   (d) Does this linearization around $\bar{x} = (0,0)$ have a Lyapunov function $V(x)$ such that $\dot{V}(x) = -2(x_1^2 + x_2^2)$. If yes, determine $V(x)$. If no, then prove that there is no such $V(x)$.

2. Consider the cost and boundary conditions

$$\int_0^1 x^2(t) + \dot{x}^2(t) + 4tx(t)\,\mathrm{d}t, \qquad x(0) = 0,\ x(1) = 2.$$

   (a) Determine the function $x_*(t)$ that minimizes this cost function with the given boundary conditions.

   (b) Is the solution found in (a) a global optimal solution?

3. Consider the system with bounded controls

$$\dot{x}_1(t) = x_2(t) \qquad x_1(0) = 0,$$
$$\dot{x}_2(t) = -x_1(t) + u(t) \qquad x_2(0) = 0, \qquad u(t) \in [-1, 1].$$

   Suppose the final time is $T = 4$ and consider the cost

$$-\tfrac{1}{2}x_1^2(4).$$

   (a) Determine the Hamiltonian, and determine the Hamiltonian equations for the costate p_1, p_2.

   (b) Show that

$$p_1(t) = A\cos(t - T) + B\sin(t - T)$$

   satisfies the Hamiltonian equations, and express the values of A and B in terms of x_1, x_2.

   (c) Determine an optimal input $u_*(t)$ explicitly as a function of time.

   (d) Is the optimal input unique?

4. Consider the system $\dot{x}(t) = x(t) + u(t), x(0) = x_0 \in \mathbb{R}$ with $u(t) \in \mathbb{R}$, and cost

$$\int_0^T \tfrac{1}{2} R u^2(t)\,\mathrm{d}t + \tfrac{1}{2} x^2(T),$$

for some $R > 0$ and $T > 0$.

(a) Derive the Riccati differential equation

(b) Verify that the solution of the Riccati differential equation is

$$P(t) = \frac{2R\mathrm{e}^{2(T-t)}}{\mathrm{e}^{2(T-t)} + 2R - 1}.$$

(c) Determine the optimal cost.

(d) Now let $T = \infty$ and discard the final cost, that is, let $J = \int_0^\infty \frac{1}{2} R u^2(t)\,\mathrm{d}t$. Determine the optimal input $u_*(t)$ as a function of $x_*(t)$.

5. (a) Formulate the Hamilton-Jacobi-Bellman theorem of Dynamic Programming.

(b) The Minimum Principle applied in Problem 3 has the disadvantage that it *assumes* the existence of an optimal control. Given the input found in 3(c) how can you use Dynamic Programming to verify that the input is indeed optimal? (You do not have to do the calculation, just explain the idea.)

problem:	1	2	3	4	5
points:	2+1+2+3	4+2	3+2+2+1	2+3+2+3	2+2

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F - \dot{x}^{\mathrm{T}}\left(\frac{\partial F}{\partial \dot{x}}\right) = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u)}{\partial p}, \; x(0) = x_0 \quad \& \quad \dot{p} = -\frac{\partial H(x,p,u)}{\partial x}, \; p(T) = \frac{\partial K(x(T))}{\partial x}$

HJB eqn: $\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}}\left[\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right] = 0, \quad V(x,T) = K(x)$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \quad P(T) = S$